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Andrews-Gordon style identities

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Abstract

In a recent paper, Griffin, Ono and Warnaar present a framework for Rogers-Ramanujan type identities using extended Hall-Littlewood polynomials $P_\lambda(x_1, x_2, \ldots; q)$ to arrive at expressions of the form

$$\sum_{\lambda:\lambda_1 \le m} q^{a|\lambda|} P_{2\lambda}(1, q, q^2, \ldots; q^n) = \text{“Infinite product modular function”}$$

for $a = 1, 2$ and any positive integers m and n. A recent paper of Rains and Warnaar presents further Rogers-Ramanujan type identities involving sums of terms $q^{|\lambda|/2} P_\lambda(1, q, q^2, \ldots; q^n)$. It is natural to attempt to reformulate these various identities to match the well-known Andrews-Gordon identities they generalize. Here, we find combinatorial formulas to replace the Hall-Littlewood polynomials and arrive at such expressions.

1 Introduction

In [3], the authors construct a general framework describing four doubly-infinite families of Rogers-Ramanujan type identities. In this context, the famous Rogers-Ramanujan identities [6]

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} \tag{1.2}$$

are presented as a special case of their Theorem 1.1 through setting the parameters $(m, n) = (1, 1)$. Fixing only $n = 1$ gives rise to the $i = 1$ and $i = m + 1$ instances of the well-known Andrews-Gordon identities [1],

$$\sum_{r_1 \ge \cdots \ge r_m \ge 0} \frac{q^{r_1^2+\ldots+r_m^2+r_i+\ldots+r_m}}{(q)_{r_1-r_2}\cdots(q)_{r_{m-1}-r_m}(q)_{r_m}} = \frac{(q^{2m+3}; q^{2m+3})_\infty}{(q)_\infty} \cdot \theta(q^i; q^{2m+3}), \tag{1.3}$$

where we use standard notation for *Pochhammer symbols*

$$(a)_k = (a; q)_k := \begin{cases} (1-a)(1-aq)\cdots(1-aq^{k-1}) & \text{if } k \ge 0 \\ \prod_{j=0}^{\infty}(1-aq^j) & \text{if } k = \infty \end{cases}$$

and *modified theta functions*

$$\theta(a;q) := (a;q)_\infty (q/a;q)_\infty.$$

For convenience, we also set

$$\theta(a_1,\ldots,a_n;q) := \theta(a_1;q)\cdots\theta(a_n;q).$$

The general identities in [3] are presented as sums over partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ of associated Hall-Littlewood polynomials in the form

$$\sum_{\lambda:\lambda_1\le m} q^{a|\lambda|} P_{2\lambda}(1,q,q^2,\ldots;q^n) = \text{“Infinite product modular function"}, \tag{1.4}$$

with $a = 1, 2$. Meanwhile, the identities in [5] take the form

$$\sum_{\lambda:\lambda_1\le m} C_\lambda q^{|\lambda|/2} P_{\lambda}(1,q,q^2,\ldots;q^n) = \text{“Infinite product modular function"}, \tag{1.5}$$

where C_λ is a particular product of Pochhammer symbols. Given this framework, it is natural to ask to what extent we can reformulate these identities to look like the Andrews-Gordon identities stated above. Here, we recast the left-hand side of these identities, without reference to partitions or Hall-Littlewood polynomials, to arrive at such explicit expressions.

The sums appearing on the left-hand sides of our identities that correspond to those in [5] range over various sequences of decreasing integers $s_1^{(j)} \ge \cdots \ge s_m^{(j)}$ related by $s_i^{(j)} \ge s_i^{(j+1)}$ for $0 \le j \le n-1$. We write $|s^{(j)}| := s_1^{(j)} + \ldots + s_m^{(j)}$ and use the convention that $s_i^{(n)} = 0$ for all i and $s_{m+1}^{(j)} = 0$ for all j. For such a collection of integers, we define

$$\begin{aligned}\mathcal{A}_{m,n}(s_*) &:= \mathcal{A}_{m,n}(s_*^{(0)}, s_*^{(1)}, \ldots, s_*^{(n)}) \\ &:= -\frac{n}{2}|s^{(0)}| + |s^{(1)}| + \ldots + |s^{(n-1)}| + \frac{n}{2}\sum_{i=1}^{m}\sum_{a=1}^{n}(s_i^{(a-1)} - s_i^{(a)})^2\end{aligned} \tag{1.6}$$

and

$$\begin{aligned}\mathcal{B}_{m,n}(s_*) &:= \mathcal{B}_{m,n}(s_*^{(0)}, s_*^{(1)}, \ldots, s_*^{(n)}; q) \\ &:= \prod_{i=1}^{m}\prod_{a=1}^{n}\frac{(q)_{s_i^{(a-1)} - s_{i+1}^{(a)}}}{(q)_{s_i^{(a-1)} - s_i^{(a)}}(q)_{s_i^{(a)} - s_{i+1}^{(a)}}}\prod_{k=1}^{m}\frac{1}{(q)_{s_k^{(0)} - s_{k+1}^{(0)}}}.\end{aligned} \tag{1.7}$$

When the identities include only Hall-Littlewood polynomials of even partitions as in (1.4), it will be more convenient to write our sums over integers $r_1 \ge \cdots \ge r_m \ge 0$ and collections $s_1^{(j)} \ge \ldots \ge s_{2m}^{(j)}$ for $1 \le j \le n-1$ satisfying $s_i^{(j)} \ge s_i^{(j+1)}$ and $r_i \ge s_{2i-1}^{(1)}$. In this case, we define

$$\mathcal{C}_{m,n}(r_*, s_*) := \mathcal{C}_{m,n}(r_*, s_*^{(1)}, \ldots, s_*^{(n)}) := \mathcal{A}_{2m,n}(s_i^{(0)} = r_{\lceil i/2\rceil}, s_*^{(1)}, \ldots, s_*^{(n)}) \tag{1.8}$$

and

$$\mathcal{D}_{m,n}(r_*, s_*; q) := \mathcal{D}_{m,n}(r_*, s_*^{(1)}, \ldots, s_*^{(n)}; q) := \mathcal{B}_{2m,n}(s_i^{(0)} = r_{\lceil i/2\rceil}, s_*^{(1)}, \ldots, s_*^{(n)}). \tag{1.9}$$

We also set $|r| := r_1 + \ldots + r_m$.

The following is a reformulation of Theorem 1.1 of [3] which more closely resembles the Andrews-Gordon identities as stated in (1.3).

Theorem 1.1. *For positive integers m and n, let $\kappa := 2m + 2n + 1$ and $n' := 2n - 1$. Then we have*

$$\sum_{r_*,s_*} \mathcal{D}_{m,n'}(r_*, s_*; q^{n'}) q^{\mathcal{C}_{m,n'}(r_*,s_*)+|r|} = \frac{(q^\kappa; q^\kappa)_\infty^n}{(q)_\infty^n} \prod_{i=1}^{n} \theta(q^{i+m}; q^\kappa) \prod_{1\le i<j\le n} \theta(q^{j-i}; q^{i+j-1}; q^\kappa)$$
$$= \frac{(q^\kappa; q^\kappa)_\infty^m}{(q)_\infty^m} \prod_{i=1}^{m} \theta(q^{i+1}; q^\kappa) \prod_{1\le i<j\le m} \theta(q^{j-i}, q^{i+j+1}; q^\kappa)$$

and

$$\sum_{r_*,s*} \mathcal{D}_{m,n'}(r_*, s_*; q^{n'}) q^{\mathcal{C}_{m,n'}(r_*,s_*)+2|r|} = \frac{(q^\kappa; q^\kappa)_\infty^n}{(q)_\infty^n} \prod_{i=1}^{n} \theta(q^{i}; q^\kappa) \prod_{1\le i<j\le n} \theta(q^{j-i}, q^{i+j}; q^\kappa)$$
$$= \frac{(q^\kappa; q^\kappa)_\infty^m}{(q)_\infty^m} \prod_{i=1}^{m} \theta(q^{i}; q^\kappa) \prod_{1\le i<j\le m} \theta(q^{j-i}, q^{i+j}; q^\kappa),$$

where the sums range over sequences of decreasing integers $r_1 \ge \cdots \ge r_m \ge 0$ and $s_1^{(j)} \ge \cdots \ge s_{2m}^{(j)} \ge 0$ for $1 \le j \le n' - 1$ satisfying $s_i^{(j)} \ge s_i^{(j+1)}$ and $r_i \ge s_{2i-1}^{(1)}$.

Remark. As promised, the Andrews-Gordon identities are easily recognized through setting $n = 1$. Since $n' = 1$, there are no s indices in the sum, giving

$$\mathcal{C}_{m,1}(r_*) = -|r| + \frac{1}{2}\sum_{i=1}^{2m} r_{\lceil i/2\rceil}^2 = -(r_1 + \ldots + r_m) + r_1^2 + \ldots + r_m^2$$

and

$$\mathcal{D}_{m,1}(r_*; q) = \prod_{j=1}^{m} \frac{1}{(q)_{r_i - r_{i+1}}} = \frac{1}{(q)_{r_1-r_2} \cdots (q)_{r_{m-1}-r_m} (q)_{r_m}}.$$

Plugging this into the two identities in the theorem above results directly in (1.3) with $i = 1$ and $i = m + 1$ respectively.

One can obtain similar reformulations of Theorems 1.2 and 1.3 of [3], which we label respectively here.

Theorem 1.2. *For positive integers m and n, let $\kappa := 2m + 2n + 2$ and $n' := 2n$. Then we have*

$$\sum_{r_*,s_*} \mathcal{D}_{m,n'}(r_*, s_*; q^{n'}) q^{\mathcal{C}_{m,n'}(r_*,s_*)+|r|}$$
$$= \frac{(q^2; q^2)_\infty \left(q^{\kappa/2}; q^{\kappa/2}\right)_\infty (q^\kappa; q^\kappa)_\infty^{n-1}}{(q)_\infty^{n+1}} \prod_{i=1}^{n} \theta\left(q^i; q^{\kappa/2}\right) \prod_{1\le i<j\le n} \theta\left(q^{j-i}, q^{i+j}; q^\kappa\right)$$
$$= \frac{(q^\kappa; q^\kappa)_\infty^m}{(q)_\infty^m} \prod_{i=1}^{m} \theta(q^{i+1}; q^\kappa) \prod_{1\le i<j\le m} \theta\left(q^{j-i}, q^{i+j+1}; q^\kappa\right).$$

Theorem 1.3. *For positive integers m and n with $n \geq 2$, let $\kappa := 2m + 2n$ and $n' := 2n$. Then we have*

$$\sum_{r_*, s_*} \mathcal{D}_{m,n'}(r_*, s_*; q^{n'}) q^{\mathcal{C}_{m,n'}(r_*, s_*)+2|r|} = \frac{(q^\kappa; q^\kappa)_\infty^n}{(q^2; q^2)_\infty (q)_\infty^{n-1}} \prod_{1 \leq i < j \leq n} \theta(q^{j-i}, q^{i+j-1}; q^\kappa)$$
$$= \frac{(q^\kappa; q^\kappa)_\infty^m}{(q)_\infty^m} \prod_{i=1}^{m} \theta(q^i; q^\kappa) \prod_{1 \leq i < j \leq m} \theta(q^{j-i}, q^{i+j}; q^\kappa).$$

We present an example of the identities that are obtained from these theorems.

Example. Setting $n = 1$ in Theorem 1.2 and writing s_i for $s_i^{(1)}$, we have that

$$\sum_{\substack{r_1 \geq \cdots \geq r_m \geq 0 \\ s_1 \geq \cdots \geq s_{2m} \geq 0 \\ r_i \geq s_{2i-1}}} \frac{q^{(r_1-s_1)^2+(r_1-s_2)^2+\ldots+(r_m-s_{2m-1})^2+(r_m-s_{2m})^2+s_1^2+\ldots+s_{2m}^2+|s|-|r|}}{(q^2;q^2)_{s_1-s_2} \cdots (q^2;q^2)_{s_{2m-1}-s_{2m}} (q^2;q^2)_{s_2} (q^2;q^2)_{r_1-r_2} \cdots (q^2;q^2)_{r_{m-1}-r_m}}$$
$$\times \frac{(q^2;q^2)_{r_1-s_3} (q^2;q^2)_{r_2-s_5} \cdots (q^2;q^2)_{r_{m-1}-s_{2m-1}}}{(q^2;q^2)_{r_1-s_1} (q^2;q^2)_{r_2-r_3} \cdots (q^2;q^2)_{r_m-s_{2m-1}}}$$

$$= \frac{(q^2;q^2)_\infty (q^{m+2}; q^{m+2})_\infty}{(q)_\infty^2} \theta(q; q^{m+2})$$
$$= \frac{(q^{2m+4}; q^{2m+4})_\infty^m}{(q)_\infty^m} \prod_{i=1}^{m} \theta(q^{i+1}; q^{2m+4}) \prod_{1 \leq i < j \leq m} \theta(q^{j-i}, q^{i+j+1}; q^{2m+4})$$

for any positive integer m. We note that the summands are all positive, viewed as formal power series in q. Specializing to $(m, n) = (2, 1)$, one finds

$$\sum_{\substack{r_1 \geq r_2 \geq 0 \\ s_1 \geq \cdots \geq s_4 \geq 0 \\ r_i \geq s_{2i-1}}} \frac{(q^2;q^2)_{r_1-s_3} q^{(r_1-s_1)^2+(r_1-s_2)^2+(r_2-s_3)^2+(r_2-s_4)^2+s_1^2+\ldots+s_4^2+s_1+\ldots+s_4-r_1-r_2}}{(q^2;q^2)_{r_1-s_1} (q^2;q^2)_{r_2-s_3} (q^2;q^2)_{s_1-s_2} (q^2;q^2)_{s_2-s_3} (q^2;q^2)_{s_3-s_4} (q^2;q^2)_{s_4} (q^2;q^2)_{r_1-r_2}}$$
$$= \frac{(q^2;q^2)_\infty (q^4;q^4)_\infty}{(q)_\infty^2} \theta(q;q^4) = \frac{(q^8;q^8)_\infty^2}{(q)_\infty^2} \theta(q^2;q^8) \theta(q^3;q^8) \theta(q, q^4; q^8).$$

We now turn to the identities of the form (1.5). The following theorems are reformulations of Theorems 5.10–5.12 of [5].

Theorem 1.4. *For positive integers m and n let $\kappa := m + 2n + 1$ and $n' := 2n$. Then we have*

$$\sum_{s_*} \mathcal{B}_{m,n'}(s_*; q^{n'}) q^{\mathcal{A}_{m,n'}(s_*)+\frac{1}{2}|s^{(0)}|}$$
$$= \frac{(q^\kappa; q^\kappa)_\infty^{n-1} (q^{\kappa/2}; q^{\kappa/2})_\infty}{(q;q)_\infty^{n-1} (q^{1/2}; q^{1/2})_\infty} \prod_{i=1}^{n} \theta(q^i; q^{\kappa/2}) \prod_{1 \leq i < j \leq n} \theta(q^{j-i}, q^{i+j}; q^\kappa),$$

where the sum ranges over sequences of decreasing integers $s_1^{(j)} \geq \cdots \geq s_m^{(j)}$ for $0 \leq j \leq n'-1$ satisfying $s_i^{(j)} \geq s_i^{(j+1)}$.

Theorem 1.5. *For positive integers m and n let $\kappa := m + 2n$ and $n' := 2n - 1$. Then we have*

$$\sum_{s_*} \mathcal{B}_{m,n'}(s_*; q^{n'}) q^{\mathcal{A}_{m,n'}(s_*)+\frac{1}{2}|s^{(0)}|}$$
$$= \frac{(q^\kappa; q^\kappa)_\infty^n}{(q;q)_\infty^{n-1}(q^{1/2};q)_\infty (q^2;q^2)_\infty} \prod_{i=1}^{n} \theta(q^{i+(m-1)/2}; q^\kappa) \prod_{1\le i<j\le n} \theta(q^{j-i}, q^{i+j-1}; q^\kappa)$$

and

$$\sum_{s_*} \mathcal{B}_{m,2n}(s_*; q^{2n}) \left(\prod_{i=1}^{m-1} (-q^n; q^n)_{s_i^{(0)}-s_{i+1}^{(0)}}\right) q^{\mathcal{A}_{m,2n}(s_*)+\frac{1}{2}|s^{(0)}|}$$
$$= \frac{(q^\kappa; q^\kappa)_\infty^{n-1}(q^{\kappa/2}; q^{\kappa/2})_\infty}{(q;q)_\infty^{n-1}(q^{1/2};q)_\infty^2 (q^2;q^2)_\infty} \prod_{i=1}^{n} \theta(q^{i-1/2}; q^{\kappa/2}) \prod_{1\le i<j\le n} \theta(q^{j-i}, q^{i+j-1}; q^\kappa).$$

Theorem 1.6. *For positive integers m and n let $\kappa := m + 2n - 1$ and $n' := 2n - 1$. Then we have*

$$\sum_{s_*} \mathcal{B}_{m,n'}(s_*; q^{n'}) \left(\prod_{i=1}^{m-1} (-q^{n-1/2}; q^{n-1/2})_{s_i^{(0)}-s_{i+1}^{(0)}}\right) q^{\mathcal{A}_{m,n'}(s_*)+\frac{1}{2}|s^{(0)}|}$$
$$= \frac{(q^\kappa;q^\kappa)_\infty^n}{(q;q)_\infty^{n-1}(q^{1/2};q^{1/2})_\infty} \prod_{i=1}^{n} \theta(q^{i+m/2-1/2}; q^\kappa) \prod_{1\le i<j\le n} \theta(q^{j-i}, q^{i+j-2}; q^\kappa).$$

Remark. As the authors mention in [5], setting $n = 1$ in Theorem 1.6 gives rise to Bressoud's even-modulus identities in [2]. Indeed, this is easily recognizable using the parameters defined in (1.6) and (1.7). Writing s_i for $s_i^{(0)}$, we have

$$\mathcal{A}_{m,1}(s_*) = -\frac{1}{2}(s_1 + \ldots + s_m) + \frac{1}{2}(s_1^2 + \ldots + s_m^2)$$

and

$$\mathcal{B}_{m,1}(s_*; q) = \prod_{j=1}^{m} \frac{1}{(q)_{s_i - s_{i+1}}}.$$

Thus, the left hand side above is

$$\sum_{s_1\ge\cdots\ge s_m\ge 0} \frac{1}{(q;q)_{s_m}} \prod_{i=1}^{m-1} \frac{(-q^{1/2};q^{1/2})_{s_i-s_{i+1}}}{(q;q)_{s_i-s_{i+1}}} q^{\frac{1}{2}(s_1^2+\ldots+s_m^2)}.$$

Putting q^2 for q and using the fact that $\frac{(-q;q)_k}{(q^2;q^2)_k} = \frac{1}{(q;q)_k}$, we obtain

$$\sum_{s_1\ge\cdots\ge s_m\ge 0} \frac{q^{s_1^2+\ldots+s_m^2}}{(q)_{s_1-s_2}\cdots(q)_{s_{m-1}-s_m}(q^2;q^2)_{s_m}} = \frac{q^{2m+2};q^{2m+2}}{(q)_\infty}\theta(q^{m+1}; q^{2m+2}).$$

These are the corresponding even-modulus identities to the odd-modulus Andrews-Gordon identities.

For integers $s_1, \ldots, s_m$ we write $\mathrm{alt}(s_*) := s_1 - s_2 + \ldots + (-1)^{m-1} s_m$ and define $d(s_*, i) := \left\lceil \frac{s_i^{(0)} - s_{i+1}^{(0)}}{2} \right\rceil$. The following is a reformulation of Theorem 5.14 of [5].

Theorem 1.7. *For positive integers m and n, let $\kappa := 2m + 2n$. Then we have*

$$\sum_{s_*}{}' \mathcal{B}_{2m,2n}(s_*) \left(\prod_{i=1}^{2m-1} (q^{2n}; q^{4n})_{d(s_*,i)} \right) q^{\mathcal{A}_{2m,2n}(s_*) + \frac{1}{2}|s^{(0)}| + \mathrm{alt}(s_*^{(0)})}$$
$$= \frac{(q^\kappa; q^\kappa)_\infty^n (-q^{\kappa/2}; q^\kappa)_\infty}{2(q;q)_\infty^n} \prod_{i=1}^{n} \theta(-q^{i-1}, q^{i+\kappa/2-1}; q^\kappa) \prod_{1 \le i < j \le n} \theta(q^{j-i}, q^{i+j-2}; q^\kappa),$$

where the prime on the sum denotes the restriction "$s_i^{(0)} - s_{i+1}^{(0)}$ is even for $i = 1, 3, \ldots, 2m-1$."

This paper is organized as follows. In the next section we define the Hall-Littlewood polynomials and recall a key formula from [4, 7]. This allows us to prove two lemmas re-expressing the Hall-Littlewood polynomials that appear in (1.4) and (1.5). We then apply these to prove Theorems 1.1–1.7 in the following section.

2 Hall-Littlewood *q*-series

Our proofs of Theorems 1.1–1.7 rely on explicit combinatorial formulas for the Hall-Littlewood polynomials appearing on the left-hand side of the identities in [3] and [5]. After defining these objects, we state and prove these two formulas as Lemmas 2.1 and 2.2.

A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a decreasing sequence of non-negative integers $\lambda_1 \ge \lambda_2 \ge \cdots$ with a finite number $l(\lambda)$ of nonzero terms. By 2λ we mean the partition $(2\lambda_1, 2\lambda_2, \ldots)$. To each partition, one can associate a Ferrers-Young diagram whose ith row consists of λ_i boxes. The *conjugate partition* λ' is defined to be the partition associated to the transpose of the Ferrers-Young diagram of λ. The multiplicity $m_i = m_i(\lambda)$ of an integer i is the number of times it appears in the partition and is equal to $\lambda'_{i+1} - \lambda'_i$. Given two partitions λ, μ we write $\lambda \subseteq \mu$ if the Ferrers-Young diagram for λ is contained in that of μ, in other words if $\lambda_i \le \mu_i$ for all i. Given a partition λ, with $\lambda_1 \le n$, the associated *Hall-Littlewood polynomial* is defined as

$$P_\lambda(x_1, \ldots, x_n; q) = \prod_{i=0}^{n} \frac{(1-q)^{m_i}}{(q)_{m_i}} \sum_{w \in \mathfrak{S}_n} w \left(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - q x_j}{x_i - x_j} \right),$$

where $m_0 := n - l(\lambda)$ and the symmetric group $\mathfrak{S}_n$ acts by permuting the x_i. One can extend this definition to symmetric functions in countably many variables as follows. If $p_r = x_1^r + x_2^2 + \ldots$ is the r-th power sum and $p_\lambda = \prod_{i \ge 1} p_{\lambda_i}$, then the set $\{p_\lambda\}$ forms a $\mathbb{Q}$-basis for the ring Λ of symmetric functions over $\mathbb{Q}$. Let $\phi_q : \Lambda[[q]] \to \Lambda[[q]]$ be the ring homomorphism determined by $\phi_q(p_r) = p_r/(1-q^r)$. Then we define the *modified Hall-Littlewood polynomials* by $P'_\lambda := \phi_q(P_\lambda)$ and

$$Q'_\lambda(x_1, \ldots, x_n; q) := P'_\lambda(x_1, \ldots, x_n; q) \prod_{i \ge 1} (q)_{\lambda'_i - \lambda'_{i+1}}. \tag{2.1}$$

From the fact that

$$\phi_{q^n}(p_r(1, q, \ldots, q^{n-1})) = \frac{1-q^{nr}}{1-q^r} \cdot \frac{1}{1-q^{nr}} = p_r(1, q, q^2, \ldots),$$

we see that

$$P_\lambda(1, q, q^2, \ldots; q^n) = P'_\lambda(1, q, \ldots, q^{n-1}; q^n). \tag{2.2}$$

We recall the following combinatorial formula for the modified Hall-Littlewood polynomials [4, 7],

$$Q'_\lambda(x_1,\dots,x_n;q) = \sum \prod_{i=1}^{\lambda_1}\prod_{a=1}^{n} x_a^{\mu_i^{(a-1)}-\mu_i^{(a)}} q^{\binom{\mu_i^{(a-1)}-\mu_i^{(a)}}{2}} \frac{(q)_{\mu_i^{(a-1)}-\mu_{i+1}^{(a)}}}{(q)_{\mu_i^{(a-1)}-\mu_i^{(a)}}(q)_{\mu_i^{(a)}-\mu_{i+1}^{(a)}}}, \tag{2.3}$$

where the sum is over partitions $0=\mu^{(n)}\subseteq\cdots\subseteq\mu^{(1)}\subseteq\mu^{(0)}=\lambda'$. We will combine (2.1)-(2.3) to arrive at the following expression for the Hall-Littlewood polynomials appearing in the sum sides of the identities we wish to rewrite.

Lemma 2.1. *Given a positive integer m and a partition λ with $\lambda_1\le m$, let $s_i^{(0)}=\lambda'_i$. Then for any positive integer n, we have*

$$P_\lambda(1,q,q^2,\dots;q^n) = \sum_{\substack{s_1^{(j)}\ge\cdots\ge s_m^{(j)}\\ s_i^{(j)}\ge s_i^{(j+1)}}} \mathcal{B}_{m,n}(s_*;q^n)q^{\mathcal{A}_{m,n}(s_*)},$$

where the sum ranges over decreasing sets of integers $s_i^{(j)}$ with $1\le j\le n-1$ and $\mathcal{A}_{m,n}(s_*)$ and $\mathcal{B}_{m,n}(s_*;q)$ are defined in (1.6) and (1.7).

Proof. For convenience, let $s_i^{(n)}=0$ and $s_{m+1}^{(j)}=0$ for all i and j. The conditions on the indices $s_i^{(j)}$ are equivalent to the condition that the partitions $\mu^{(j)}$ defined by $\mu^{(j)}=(s_1^{(j)},\dots,s_m^{(j)},0,0,\dots)$ satisfy $0=\mu^{(n)}\subseteq\cdots\subseteq\mu^{(1)}\subseteq\mu^{(0)}=\lambda'$. Thus, from (2.3) we have

$$Q'_\lambda(1,q,\dots,q^{n-1};q^n) = \sum_{\substack{s_1^{(j)}\ge\cdots\ge s_m^{(j)}\\ s_\ell^{(j)}\ge s_\ell^{(j+1)}}} \prod_{i=1}^{m}\prod_{a=1}^{n} q^{(a-1)(s_i^{(a-1)}-s_i^{(a)})} q^{n\binom{s_i^{(a-1)}-s_i^{(a)}}{2}} \times \frac{(q^n;q^n)_{s_i^{(a-1)}-s_{i+1}^{(a)}}}{(q^n;q^n)_{s_i^{(a-1)}-s_i^{(a)}}(q^n;q^n)_{s_i^{(a)}-s_{i+1}^{(a)}}}.$$

Recall that we write $|s^{(j)}|=s_1^{(j)}+\ldots+s_m^{(j)}$. The power of q appearing in a term of the sum corresponding to an index set $s_i^{(j)}$ is given by

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{a=1}^{n}\left[(a-1)(s_i^{(a-1)}-s_i^{(a)})+n\binom{s_i^{(a-1)}-s_i^{(a)}}{2}\right]\\
&\quad=\sum_{i=1}^{m}\sum_{a=1}^{n}\left(a-1-\frac{n}{2}\right)(s_i^{(a-1)}-s_i^{(a)})+\frac{n}{2}\sum_{i=1}^{m}\sum_{a=1}^{n}(s_i^{(a-1)}-s_i^{(a)})^2\\
&\quad=\sum_{a=1}^{n}\left(a-1-\frac{n}{2}\right)(|s^{(a-1)}|-|s^{(a)}|)+\frac{n}{2}\sum_{i=1}^{m}\sum_{a=1}^{n}(s_i^{(a-1)}-s_i^{(a)})^2\\
&\quad=-\frac{n}{2}|s^{(0)}|+|s^{(1)}|+\ldots+|s^{(n-1)}|+\frac{n}{2}\sum_{i=1}^{m}\sum_{a=1}^{n}(s_i^{(a-1)}-s_i^{(a)})^2\\
&\quad=\mathcal{A}_{m,n}(s_*).
\end{aligned}$$

In addition, this power of q is multiplied the following product of Pochhammer symbols

$$\prod_{i=1}^{m}\prod_{a=1}^{n}\frac{(q^n;q^n)_{s_i^{(a-1)}-s_{i+1}^{(a)}}}{(q^n;q^n)_{s_i^{(a-1)}-s_i^{(a)}}(q^n;q^n)_{s_i^{(a)}-s_{i+1}^{(a)}}}=\mathcal{B}(s_*;q^n)\prod_{k=1}^{m}(q^n;q^n)_{s_k^{(0)}-s_{k+1}^{(0)}}.$$

Thus, we have

$$\begin{aligned} Q'_\lambda(1,q,\ldots,q^{n-1};q^n) &= \sum_{\substack{s_1^{(j)}\ge\cdots\ge s_m^{(j)}\\ s_i^{(j)}\ge s_i^{(j+1)}}}\prod_{k=1}^{m}(q^n;q^n)_{s_k^{(0)}-s_{k+1}^{(0)}}\mathcal{B}_{m,n}(s_*;q^n)q^{\mathcal{A}_{m,n}(s_*)}\\ &= \prod_{k=1}^{m}(q^n;q^n)_{s_k^{(0)}-s_{k+1}^{(0)}}\sum_{\substack{s_1^{(j)}\ge\cdots\ge s_m^{(j)}\\ s_i^{(j)}\ge s_i^{(j+1)}}}\mathcal{B}_{m,n}(s_*;q^n)q^{\mathcal{A}_{m,n}(s_*)}, \end{aligned}$$

so using (2.2) and (2.1), we may write

$$\begin{aligned} P_\lambda(1,q,q^2,\ldots;q^n) &= P'_\lambda(1,q,\ldots,q^{n-1};q^n)\\ &= \frac{Q'_\lambda(1,q,\ldots,q^{n-1};q^n)}{\prod_{k=1}^{m}(q^n;q^n)_{s_k^{(0)}-s_{k+1}^{(0)}}}\\ &= \sum_{\substack{s_1^{(j)}\ge\cdots\ge s_m^{(j)}\\ s_i^{(j)}\ge s_i^{(j+1)}}}\mathcal{B}_{m,n}(r_*,s_*;q^n)q^{\mathcal{A}_{m,n}(r_*,s_*)}. \end{aligned}$$

□

We also provide the following formula for Hall-Littlewood polynomials of even partitions.

Lemma 2.2. *Given a positive integer m and a partition λ with $\lambda_1\le m$, let $r_i=\lambda'_i$. Then for any positive integer n, we have*

$$P_{2\lambda}(1,q,q^2,\ldots;q^n)=\sum_{\substack{s_1^{(j)}\ge\cdots\ge s_{2m}^{(j)}\\ r_i\ge s_{2i-1}^{(1)},s_i^{(j)}\ge s_i^{(j+1)}}}\mathcal{D}_{m,n}(r_*,s_*;q^n)q^{\mathcal{C}_{m,n}(r_*,s_*)},$$

where the sum ranges over collections of decreasing integers $s_i^{(j)}$ for $1\le j\le n-1$ and $\mathcal{C}_{m,n}(r_*,s_*)$ and $\mathcal{D}_{m,n}(r_*,s_*;q)$ are defined in (1.8) and (1.9).

Proof. Applying the previous lemma to the partition 2λ and recalling the definitions of $\mathcal{C}_{m,n}(r_*,s_*)$ and $\mathcal{D}_{m,n}(r_*,s_*;q)$ results directly in this expression. □

3 Proof of theorems

The proofs of Theorems 1.1–1.3 follow immediately from Lemma 2.2 and the respectively labeled theorems of [3]. A sum over all partitions λ with $\lambda_1\le m$ is the same as a sum over all partitions whose conjugates have length $l(\lambda')\le m$. We may represent these partitions by their conjugates, which are specified by indices $r_1 \geq \cdots \geq r_m \geq 0$. This shows in each case that

$$\sum_{r_*, s_*} \mathcal{D}_{m,n'}(r_*, s_*; q^{n'}) q^{\mathcal{C}_{m,n'}(r_*, s_*) + a|r|} = \sum_{\lambda:\lambda_1 \leq m} q^{a|\lambda|} P_{2\lambda}(1, q, q^2, \ldots; q^{n'})$$
$$= \text{“Infinite product modular function"}$$

for $a = 1, 2$.

The proofs of Theorems 1.4–1.7 follow from Theorems 5.10–5.12, 5.14 of [5] respectively by rewriting the sums that appear there in a similar way. In this case, we represent the sum over all partitions λ with $\lambda_1 \leq m$ as a sum over their conjugates, which we specify by $s_1^{(0)} \geq \cdots \geq s_m^{(0)}$, and use Lemma 2.1 to rewrite the Hall-Littlewood polynomials. We note that with this notation we have $|\lambda| = s_1^{(0)} + \ldots + s_m^{(0)} = |s^{(0)}|$ and $m_i(\lambda) = s_i^{(0)} - s_{i+1}^{(0)}$. In addition, the number of odd parts of λ, written as odd(λ) in Theorem 5.14 of [5], is equal to

$$\sum_{i \text{ odd}} m_i(\lambda) = \sum_{i \text{ odd}} s_i^{(0)} - s_{i+1}^{(0)} = s_1^{(0)} - s_2^{(0)} + \ldots + (-1)^{m-1} s_m^{(0)} = \mathrm{alt}(s_*^{(0)}).$$

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